

RGPVNOTES.IN

Program : **B.Tech**

Subject Name: **Discrete Structure**

Subject Code: **IT-302**

Semester: **3rd**



LIKE & FOLLOW US ON FACEBOOK

facebook.com/rgpvnotes.in

Subject Notes Discrete Structures

UNIT-5

Posets, Hasse Diagram and Lattices:

Introduction: Elementary mathematics tends to focus lopsidedly on computational structures. Relational ideas have become more important with the advent of computer science and the rise of discrete mathematics, however. Many contemporary mathematical applications involve binary or n-ary relations in addition to computations. In this chapter we will explore other kinds of relations (these will all be binary relations here), particularly ones that impose an order of one sort or another on a set. This will lead us to investigate certain order-structures (posets, lattices) and to introduce an abstract type of algebra known as Boolean algebra. Our exploration of these ideas will nicely tie together some earlier ideas in logic and set theory as well as lead us into areas that are of crucial importance to computer science.

Ordered set - If a set X is ordered in a reasonable way, then there is a natural way to define an "order topology" on X . Most interesting (for our purposes) will be ordered sets that satisfy a very strong ordering condition: that every nonempty subset contains a smallest element. Such sets are called well-ordered. The most familiar example of a well-ordered set is \mathbb{N} .

Partially Ordered Set (POSET) - A partially ordered set consists of a set with a binary relation which is reflexive, antisymmetric and transitive. "Partially ordered set" is abbreviated as POSET.

Examples

1. The set of real numbers under binary operation less than or equal to (\leq) is a poset.

Solution - Let the set $S = \{1, 2, 3\}$ and the operation is \leq

The relations will be $\{(1,1), (2,2), (3,3), (1,2), (1,3), (2,3)\}$

This relation R is **reflexive** as $\{(1, 1), (2, 2), (3, 3)\} \in R$

This relation R is **antisymmetric**, as $\{(1, 2), (1,3), (2,3)\} \in R$ and $\{(1,2), (1,3), (2,3)\} \notin R$

This relation R is also **transitive** as $\{(1, 2), (2, 3), (1, 3)\} \in R$

Hence, it is a **poset**.

Note: The vertex set of a directed acyclic graph under the operation 'reachability' is a poset.

Hasse Diagram of Partially Ordered Set - The Hasse diagram of a poset is the directed graph whose vertices are the element of that poset and the arcs covers the pairs (x, y) in the poset. If in the poset $x < y$, then the point x appears lower than the point y in the Hasse diagram. If $x < y < z$ in the poset, then the arrow is not shown between x and z as it is implicit.

Example

The poset of subsets of $\{1,2,3\} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ is shown by the following Hasse diagram -

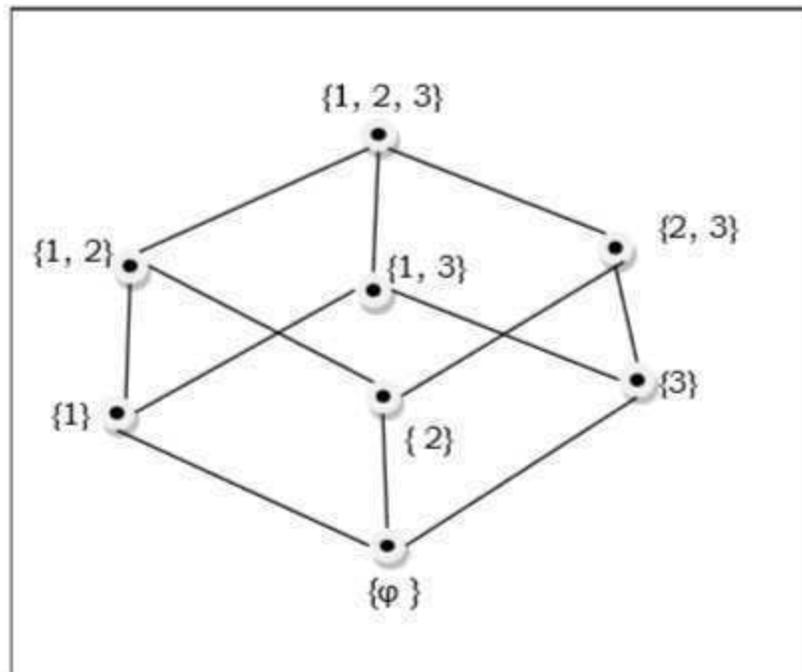


Figure 5.1 Hasse diagram

Isomorphic ordered set - Two partially ordered sets are said to be isomorphic if their "structures" are entirely analogous. Formally, partially ordered sets $P=(X, \leq)$ and $Q=(X', \leq')$ are isomorphic if there is a bijection f from X to X' such that $x_1 \leq x_2$ precisely when $f(x_1) \leq' f(x_2)$.

Well-ordered set- A well-ordered set is a totally ordered set in which every nonempty subset has a least member.

An example of well-ordered set is the set of positive integers with the standard order relation (\mathbb{Z}^+) because any nonempty subset of it has least member. However, \mathbb{R}^+ (the positive reals) is not a well-ordered set with the usual order, because $(0, 1) = \{x: 0 < x < 1\}$ is a nonempty subset but it doesn't contain at least number.

A well-ordering of a set X is the result of defining a binary relation \leq on X to itself in such a way that X becomes well-ordered with respect to \leq .

Linearly Ordered Set - A Linearly ordered set or Total ordered set is a partial order set in which every pair of element is comparable. The elements $a, b \in S$ are said to be comparable if either $a \leq b$ or $b \leq a$ holds. Trichotomy law defines this total ordered set. A totally ordered set can be defined as a distributive lattice having the property $\{a \vee b, a \wedge b\} = \{a, b\}$ for all values of a, b in set S .

Lattice - A lattice is a poset (L, \leq) for which every pair $\{a, b\} \in L$ has a least upper bound (denoted by $a \vee b$) and a greatest lower bound (denoted by $a \wedge b$). LUB $(\{a, b\})$ is called the join of a and b . GLB $(\{a, b\})$ is called the meet of a and b .

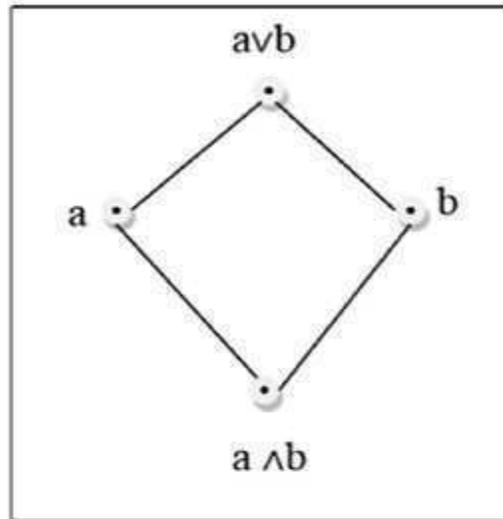


Figure 5.2 Lattice

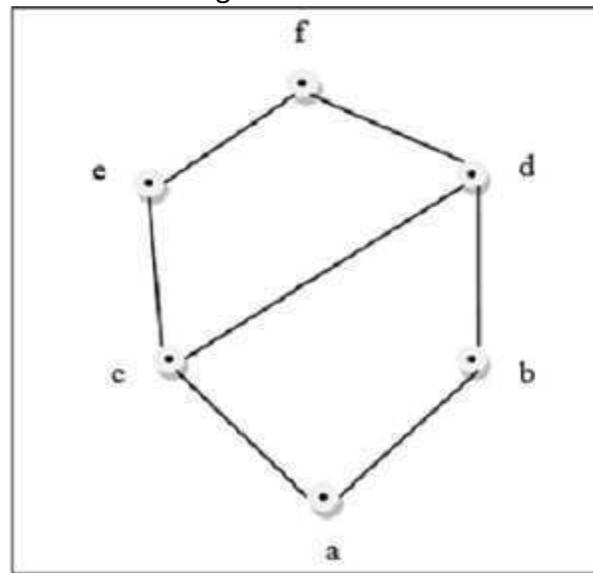


Figure 5.3 example of lattice

Figure 5.3 is a lattice because for every pair $\{a, b\} \in L$, a GLB and a LUB exists.

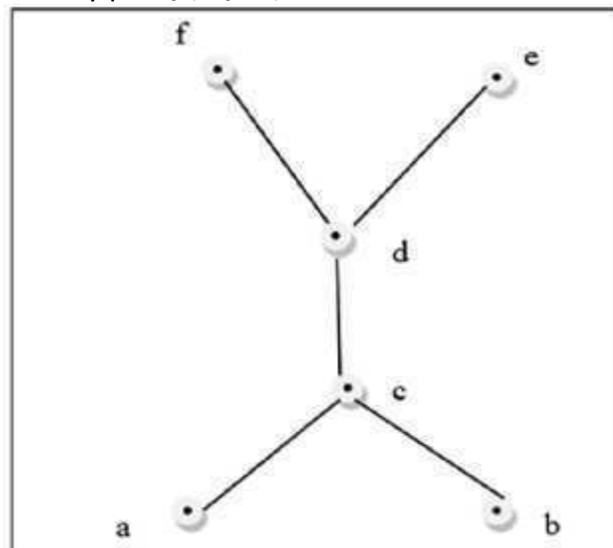


Figure 5.4 example of lattice

Figure 5.4 is a not a lattice because $GLB(a, b)$ and $LUB(e, f)$ does not exist.

Properties of Lattices

1. Idempotent Properties

$$a) a \vee a = a$$

$$b) a \wedge a = a$$

2. Absorption Properties

$$a) a \vee (a \wedge b) = a$$

$$b) a \wedge (a \vee b) = a$$

3. Commutative Properties

$$a) a \vee b = b \vee a$$

$$b) a \wedge b = b \wedge a$$

4. Associative Properties

$$a) a \vee (b \vee c) = (a \vee b) \vee c$$

$$b) a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

Dual of a Lattice - The dual of a lattice is obtained by interchanging the ' \vee ' and ' \wedge ' operations.

Example

The dual of $[a \vee (b \wedge c)]$ is $[a \wedge (b \vee c)]$

Types of Lattices

1. **Bounded Lattice** - A lattice L becomes a bounded lattice if it has a greatest element 1 and a least element 0 .

2. **Complemented Lattice** - A lattice L becomes a complemented lattice if it is a bounded lattice and if every element in the lattice has a complement. An element x has a complement x' if $\exists x(x \wedge x' = 0 \text{ and } x \vee x' = 1)$

3. **Distributive Lattice** - If a lattice satisfies the following two distribute properties, it is called a distributive lattice.

$$a) a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$b) a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

4. **Modular Lattice** - If a lattice satisfies the following property, it is called modular lattice.

$$a \wedge (b \vee (a \wedge d)) = (a \wedge b) \vee (a \wedge d)$$

5. **Isomorphic Lattices** If $f: L_1 \rightarrow L_2$ is an isomorphism from the poset (L_1, \leq_1) to the poset (L_2, \leq_2) then L_1 is a lattice iff L_2 is a lattice. If a and b are elements of L_1 then $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$ If two lattices are isomorphic as posets we say they are isomorphic lattices.

Combinatorics: Combinatorics is the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties. Mathematicians sometimes use the term "combinatorics" to refer to a larger subset of discrete mathematics that includes graph theory.

Counting Theory - In daily lives, many a times one needs to find out the number of all possible outcomes for a series of events. For instance, in how many ways can a panel of judges comprising of 6 men and 4 women be chosen from among 50 men and 38 women? How many different 10 lettered PAN numbers can be generated such that the first five letters are capital alphabets, the next four are digits and the last is again a capital letter. For solving these problems, mathematical theory of counting are used. **Counting** mainly encompasses fundamental counting rule, the permutation rule, and the combination rule.

The Rules of Sum and Product

The **Rule of Sum** and **Rule of Product** are used to decompose difficult counting problems into simple problems.

1. **The Rule of Sum** - If a sequence of tasks T_1, T_2, \dots, T_m can be done in w_1, w_2, \dots, w_m ways respectively (the condition is that no tasks can be performed simultaneously), then the number of ways to do one

of these tasks is $w_1+w_2+\dots+w_m$. If we consider two tasks A and B which are disjoint (i.e. $A \cap B = \emptyset$), then mathematically $|A \cup B| = |A| + |B|$

2. **The Rule of Product** – If a sequence of tasks T_1, T_2, \dots, T_m can be done in w_1, w_2, \dots, w_m ways respectively and every task arrives after the occurrence of the previous task, then there are $w_1 \times w_2 \times \dots \times w_m$ ways to perform the tasks. Mathematically, if a task B arrives after a task A, then $|A \times B| = |A| \times |B|$

Example

Question – A boy lives at X and wants to go to School at Z. From his home X he has to first reach Y and then Y to Z. He may go X to Y by either 3 bus routes or 2 train routes. From there, he can either choose 4 bus routes or 5 train routes to reach Z. How many ways are there to go from X to Z?

Solution – From X to Y, he can go in $3+2=5$ ways (Rule of Sum). Thereafter, he can go Y to Z in $4+5=9$ ways (Rule of Sum). Hence from X to Z he can go in $5 \times 9 = 45$ ways (Rule of Product).

Permutations - A permutation is an arrangement of some elements in which order matters. In other words a Permutation is an ordered Combination of elements.

Examples

1. From a set $S = \{x, y, z\}$ by taking two at a time, all permutations are – xy, yx, xz, zx, yz, zy
2. We have to form a permutation of three digit numbers from a set of numbers $S = \{1, 2, 3\}$

Different three digit numbers will be formed when we arrange the digits. The permutation will be = 123, 132, 213, 231, 312, 321

Number of Permutations

The number of permutations of 'n' different things taken 'r' at a time is denoted by ${}^n P_r$

$${}^n P_r = \frac{n!}{(n-r)!} \text{ Where } n! = 1.2.3 \dots (n-1).n$$

Proof – Let there be 'n' different elements.

There is n number of ways to fill up the first place. After filling the first place (n-1) number of elements is left. Hence, there are (n-1) ways to fill up the second place. After filling the first and second place, (n-2) number of elements is left. Hence, there are (n-2) ways to fill up the third place. We can now generalize the number of ways to fill up r-th place as $[n - (r-1)] = n-r+1$

So, the total no. of ways to fill up from first place up to r-th-place –

$${}^n P_r = n(n-1)(n-2) \dots (n-r+1)$$

$$= \frac{[n(n-1)(n-2) \dots (n-r+1)][(n-r)(n-r-1) \dots 3.2.1]}{[(n-r)(n-r-1) \dots 3.2.1]}$$

Hence,

$${}^n P_r = \frac{n!}{(n-r)!}$$

Some important formulas of permutation

1. If there are n elements of which a_1 are alike of some kind, a_2 are alike of another kind; a_3 are alike of third kind and so on and a_r are of rth kind, where $(a_1+a_2+\dots+a_r)=n$ Then, number of permutations of these n objects is = $\frac{n!}{(a_1!(a_2!) \dots (a_r!)}$
2. Number of permutations of n distinct elements taking n elements at a time = ${}^n P_n = n!$
3. The number of permutations of n dissimilar elements taking r elements at a time, when x particular things always occupy definite places = ${}^{n-x} P_{r-x}$
4. The number of permutations of n dissimilar elements when r specified things always come together is – $r! (n-r+1)!$
5. The number of permutations of n dissimilar elements when r specified things never come together is – $n! - [r! (n-r+1)!]$
6. The number of circular permutations of n different elements taken x elements at time = ${}^n P_x / x$
7. The number of circular permutations of n different things = ${}^n P_n / n$

Some Problems

Problem 1 – from a bunch of 6 different cards, how many ways we can permute it?

Solution – as we are taking 6 cards at a time from a deck of 6 cards, the permutation will be ${}^6P_6=6!=720$

Problem 2 – in how many ways can the letters of the word 'READER' be arranged?

Solution – There are 6 letters word (2 E, 1 A, 1D and 2R.) in the word 'READER'.

The permutation will be $=6! / [(2!)(1!)(1!)(2!)] = 180$.

Problem 3 – in how ways can the letters of the word 'ORANGE' be arranged so that the consonants occupy only the even positions?

Solution – There are 3 vowels and 3 consonants in the word 'ORANGE'. Number of ways of arranging the consonants among themselves $={}^3P_3=3!=6$.

The remaining 3 vacant places will be filled up by 3 vowels in ${}^3P_3=3!=6$ ways. Hence, the total number of permutation is $6 \times 6 = 36$

Combinations – A combination is selection of some given elements in which order does not matter. The number of all combinations of n things, taken r at a time is –

$${}^nC_r = n! / (r! (n - r)!)$$

Problem 1 - Find the number of subsets of the set {1, 2, 3, 4, 5, 6} having 3 elements.

Solution - The cardinality of the set is 6 and we have to choose 3 elements from the set. Here, the ordering does not matter. Hence, the number of subsets will be ${}^6C_3=20$

Problem 2 - There are 6 men and 5 women in a room. In how many ways we can choose 3 men and 2 women from the room?

Solution - The number of ways to choose 3 men from 6 men is 6C_3 and the number of ways to choose 2 women from 5 women is 5C_2

Hence, the total number of ways is $-{}^6C_3 \times {}^5C_2 = 20 \times 10 = 200$

Problem 3 - How many ways can you choose 3 distinct groups of 3 students from total 9 students?

Solution - Let us number the groups as 1, 2 and 3

For choosing 3 students for 1st group, the number of ways $-{}^9C_3$

The number of ways for choosing 3 students for 2nd group after choosing 1st group $-{}^6C_3$

The number of ways for choosing 3 students for 3rd group after choosing 1st and 2nd group $-{}^3C_3$

Hence, the total number of ways $={}^9C_3 \times {}^6C_3 \times {}^3C_3 = 84 \times 20 \times 1 = 1680$

Pascal's Identity

Pascal's identity, first derived by Blaise Pascal in 19th century, states that the number of ways to choose k elements from n elements is equal to the summation of number of ways to choose (k-1) elements from (n-1) elements and the number of ways to choose elements from n-1 elements.

Mathematically, for any positive integers k and n: ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

Proof –

$${}^{n-1}C_{k-1} + {}^{n-1}C_k$$

$$= (n-1)! (k-1)! (n-k)! + (n-1)! k! (n-k-1)!$$

$$= (n-1)! (kk!(n-k)! + n-kk!(n-k)!)$$

$$= (n-1)! nk!(n-k)!$$

$$= n! k! (n-k)!$$

$$= nC_k$$

Probability - Closely related to the concepts of counting is Probability. We often try to guess the results of games of chance, like card games, slot machines, and lotteries; i.e. we try to find the likelihood or probability that a particular result will be obtained.

Probability can be conceptualized as finding the chance of occurrence of an event. Mathematically, it is the study of random processes and their outcomes. The laws of probability have a wide applicability in a variety of fields like genetics, weather forecasting, opinion polls, stock markets etc.

Basic Concepts

Probability theory was invented in the 17th century by two French mathematicians, Blaise Pascal and Pierre de Fermat, who were dealing with mathematical problems regarding of chance.

Before proceeding to details of probability, let us get the concept of some definitions.

Random Experiment – an experiment in which all possible outcomes are known and the exact output cannot be predicted in advance is called a random experiment. Tossing a fair coin is an example of random experiment.

Sample Space – When we perform an experiment, then the set S of all possible outcomes is called the sample space. If we toss a coin, the sample space $S = \{H, T\}$

Event – any subset of a sample space is called an event. After tossing a coin, getting Head on the top is an event.

The word "probability" means the chance of occurrence of a particular event. The best we can say is how likely they are to happen, using the idea of probability.

Probability of occurrence of an event = $\frac{\text{Total number of favorable outcome}}{\text{total number of outcomes}}$

As the occurrence of any event varies between 0% and 100%, the probability varies between 0 and 1.

Steps to find the probability

Step 1 – Calculate all possible outcomes of the experiment.

Step 2 – Calculate the number of favorable outcomes of the experiment.

Step 3 – Apply the corresponding probability formula.

Tossing a Coin

If a coin is tossed, there are two possible outcomes – Heads (H) or Tails (T)

So, Total number of outcomes = 2

Hence, the probability of getting a Head (H)

on top is $1/2$ and the probability of getting a Tails (T)

on top is $1/2$

Throwing a Dice

When a dice is thrown, six possible outcomes can be on the top – 1, 2, 3, 4, 5, 6.

The probability of any one of the numbers is $1/6$

The probability of getting even numbers is $3/6 = 1/2$

The probability of getting odd numbers is $3/6 = 1/2$

Taking Cards from a Deck

From a deck of 52 cards, if one card is picked find the probability of an ace being drawn and also find the probability of a diamond being drawn.

Total number of possible outcomes – 52

Outcomes of being an ace – 4

Probability of being an ace = $4/52 = 1/13$

Probability of being a diamond = $13/52 = 1/4$

Probability Axioms

1. The probability of an event always varies from 0 to 1. $[0 \leq P(x) \leq 1]$
2. For an impossible event the probability is 0 and for a certain event the probability is 1.
3. If the occurrence of one event is not influenced by another event, they are called mutually exclusive or disjoint.
4. If A_1, A_2, \dots, A_n are mutually exclusive/disjoint events, then $P(A_i \cap A_j) = \emptyset$ for $i \neq j$ and $P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$

Properties of Probability

1. If there are two events x and x^c which are complementary, then the probability of the complementary event is –

$$p(x^c) = 1 - p(x)$$

2. For two non-disjoint events A and B, the probability of the union of two events –
 $P(A \cup B) = P(A) + P(B)$

3. If an event A is a subset of another event B (i.e. $A \subset B$), then the probability of A is less than or equal to the probability of B. Hence, $A \subset B$ implies $P(A) \leq P(B)$

Conditional Probability

The conditional probability of an event B is the probability that the event will occur given an event A has already occurred. This is written as $P(B|A)$.

Mathematically – $P(B|A) = P(A \cap B) / P(A)$

If event A and B are mutually exclusive, then the conditional probability of event B after the event A will be the probability of event B that is $P(B)$

Problem 1 - In a country 50% of all teenagers own a cycle and 30% of all teenagers own a bike and cycle. What is the probability that a teenager owns bike given that the teenager owns a cycle?

Solution - Let us assume A is the event of teenagers owning only a cycle and B is the event of teenagers owning only a bike.

So, $P(A) = 50/100 = 0.5$ and $P(A \cap B) = 30/100 = 0.3$ from the given problem.

$P(B|A) = P(A \cap B) / P(A) = 0.3 / 0.5 = 0.6$

Hence, the probability that a teenager owns bike given that the teenager owns a cycle is 60%.

Problem 2 - In a class, 50% of all students play cricket and 25% of all students play cricket and volleyball. What is the probability that a student plays volleyball given that the student plays cricket?

Solution - Let us assume A is the event of students playing only cricket and B is the event of students playing only volleyball.

So, $P(A) = 50/100 = 0.5$ and $P(A \cap B) = 25/100 = 0.25$ from the given problem.

$0.25 / 0.5 = 0.5$

Hence, the probability that a student plays volleyball given that the student plays cricket is 50%.

Problem 3 - Six good laptops and three defective laptops are mixed up. To find the defective laptops all of them are tested one-by-one at random. What is the probability to find both of the defective laptops in the first two pick?

Solution- Let A is the event that we find a defective laptop in the first test and B is the event that we find a defective laptop in the second test.

Hence, $P(A \cap B) = P(A)P(B|A) = 3/9 \times 2/8 = 1/21$

Bayes' Theorem

Theorem – If A and B are two mutually exclusive events, where $P(A)$ is the probability of A and $P(B)$ is the probability of B, $P(A|B)$ is the probability of A given that B is true. $P(B|A)$ is the probability of B given that A is true, then Bayes' Theorem states –

$P(A|B) = P(B|A)P(A) / \sum_{i=1}^n P(B|A_i)P(A_i)$

Application of Bayes' Theorem

1. in situations where all the events of sample space are mutually exclusive events.

2. In situations where either $P(A_i \cap B)$ for each A_i or $P(A_i)$ and $P(B|A_i)$ for each A_i is known.

Problem - Consider three pen-stands. The first pen-stand contains 2 red pens and 3 blue pens; the second one has 3 red pens and 2 blue pens; and the third one has 4 red pens and 1 blue pen. There is equal probability of each pen-stand to be selected. If one pen is drawn at random, what is the probability that it is a red pen?

Solution - Let A_i be the event that i^{th} pen-stand is selected. Here, $i = 1, 2, 3$.

Since probability for choosing a pen-stand is equal, $P(A_i) = 1/3$

Let B be the event that a red pen is drawn.

The probability that a red pen is chosen among the five pens of the first pen-stand,

$P(B|A_1) = 2/5$

The probability that a red pen is chosen among the five pens of the second pen-stand,

$$P(B|A_2) = 3/5$$

The probability that a red pen is chosen among the five pens of the third pen-stand,

$$P(B|A_3) = 4/5$$

According to Bayes' Theorem,

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)$$

$$= 1/3 \cdot 2/5 + 1/3 \cdot 3/5 + 1/3 \cdot 4/5$$

$$= 3/5$$

Binomial Probability Distribution

Binomial Distribution It is a discrete probability distribution.

Binomial Probability is calculated by following general formula-

$$P(X) = {}^n C_x p^x q^{(n-x)}$$

Where,

n = number of trials

x = number of success

p = Probability of success

q = Probability of failure = 1 - p.

Requirements for a Binomial Distribution

- Random Experiment must involve n identical trials.
- As the word "Binomial" suggests, each trial should have only 2 possible outcomes, denoted as "Success" or "Failure".
- Each trial is independent of the previous trials.
- The probability of success denoted by p does not change from trial to trial.
- The probability of failure is 1-p and it is also fixed from trial to trial.

Sample question: "60% of people who purchase sports cars are men. If 10 sports car owners are randomly selected, find the probability that exactly 7 are men."

Step 1: Identify 'n' and 'X' from the problem.

Using our sample question, n (the number of randomly selected items — in this case, sports car owners are randomly selected) is 10, and X (the number you are asked to "find the probability" for) is 7.

Step 2: Figure out the first part of the formula, which is:

$$n! / (n - X)! X!$$

Substituting the variables:

$$10! / ((10 - 7)! \times 7!)$$

This equals to 120

Set this number aside for a moment.

Step 3: Find "p" the probability of success and "q" the probability of failure.

We are given p = 60%, or .6.

Therefore, the probability of failure is 1 - .6 = .4 (40%).

Step 4: Work the next part of the formula. $p^X = .6^7 = .0279936$ Set this number aside while you work the third part of the formula.

Step 5: Work the third part of the formula.

$$q \cdot (n - X) =$$

$$.4(10-7) =$$

$$.43 = 0.064$$

Step 6: Multiply the three answers from steps 2, 4 and 5 together.

$$120 \times 0.0279936 \times 0.064$$

$$= 0.215.$$

Recurrence Relation - It Show how recursive techniques can derive sequences and be used for solving

counting problems. The procedure for finding the terms of a sequence in a recursive manner is called recurrence relation. We study the theory of linear recurrence relations and their solutions. Finally, we introduce generating functions for solving recurrence relations.

Definition

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i < n$).

Example – Fibonacci series – $F_n = F_{n-1} + F_{n-2}$, Tower of Hanoi – $F_n = 2F_{n-1} + 1$

Recursive algorithms:

- All of the algorithms that have been developed so far have been iterative - they perform their task by repeating and following some statements in such a way that a task is achieved.
- Recursion is the process of defining something in terms of itself. Usually, the algorithm will attempt to simply the problem at hand by merely getting closer to the final result. The process is repeated until the problem has simplified enough so that it can be directly evaluated.

Example:

$$T(n) = T(n-1) + 1$$

$$T(0) = 1$$

Solving:

$$T(n-1) = T(n-2) + 1,$$

$$\text{So } T(n) = T(n-1) + 1 = [T(n-2) + 1] + 1 = T(n-2) + 2,$$

$$\text{And } T(n-2) = T(n-3) + 1$$

$$\text{So } T(n) = T(n-2) + 2 = [T(n-3) + 1] + 1 = T(n-3) + 3,$$

$$\text{And in general } T(n) = T(n-i) + i;$$

$$\text{So finally } T(n) = T(0) + n = 1 + n$$



Linear Recurrence Relations with constant coefficients

A linear recurrence equation of degree k or order k is a recurrence equation which is in the format $x_n = A_1 x_{n-1} + A_2 x_{n-2} + A_3 x_{n-3} + \dots + A_k x_{n-k}$ (A_n is a constant and $A_k \neq 0$) on a sequence of numbers as a first-degree polynomial.

These are some examples of linear recurrence equations –

Recurrence relations Initial values Solutions

$$F_n = F_{n-1} + F_{n-2} \quad a_1 = a_2 = 1 \quad \text{Fibonacci number}$$

$$F_n = F_{n-1} + F_{n-2} \quad a_1 = 1, a_2 = 3 \quad \text{Lucas Number}$$

$$F_n = F_{n-2} + F_{n-3} \quad a_1 = a_2 = a_3 = 1 \quad \text{Padovan sequence}$$

$$F_n = 2F_{n-1} + F_{n-2} \quad a_1 = 0, a_2 = 1 \quad \text{Pell number}$$

Homogeneous Recurrence Relation: A recurrence relation is called homogeneous if it is in the form

$$F_n = AF_{n-1} + BF_{n-2}$$

How to solve Homogeneous linear recurrence relation (Homogeneous solution)

Suppose, a two ordered linear recurrence relation is – $F_n = AF_{n-1} + BF_{n-2}$ where A and B are real numbers.

The characteristic equation for the above recurrence relation is –

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots –

Case 1 – If this equation factors as $(x-x_1)(x-x_2) = 0$ and it produces two distinct real roots x_1 and x_2 , then $F_n =$

$ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]

Case 2 – If this equation factors as $(x-x_1)^2=0$ and it produces single real root x_1 , then $F_n = ax_1^n + bx_2^n$ is the solution.

Case 3 – If the equation produces two distinct complex roots, x_1 and x_2 in polar form $x_1=r\angle\theta$ and $x_2=r\angle(-\theta)$, then $F_n=r^n(\text{acos}(n\theta)+b\text{sin}(n\theta))$ is the solution.

Problem 1 - Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0=1$ and $F_1=4$

Solution - The characteristic equation of the recurrence relation is –

$$x^2 - 5x + 6 = 0,$$

$$\text{So, } (x-3)(x-2) = 0$$

Hence, the roots are –

$$x_1 = 3 \text{ and } x_2 = 2$$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is –

$$F_n = ax_1^n + bx_2^n$$

$$\text{Here, } F_n = a3^n + b2^n \text{ (As } x_1=3 \text{ and } x_2=2)$$

Therefore,

$$1 = F_0 = a3^0 + b2^0 = a + b$$

$$4 = F_1 = a3^1 + b2^1 = 3a + 2b$$

Solving these two equations, we get $a=2$ and $b=-1$

Hence, the final solution is –

$$F_n = 2 \cdot 3^n + (-1) \cdot 2^n = 2 \cdot 3^n - 2^n$$

Problem 2 Solve the recurrence relation – $F_n = 10F_{n-1} - 25F_{n-2}$ where $F_0=3$ and $F_1=17$

Solution The characteristic equation of the recurrence relation is –

$$x^2 - 10x + 25 = 0$$

$$\text{So } (x-5)^2 = 0$$

Hence, there is single real root $x_1=5$

As there is single real valued root, this is in the form of case 2

Hence, the solution is –

$$F_n = ax_1^n + bx_2^n$$

$$3 = F_0 = a \cdot 5^0 + b \cdot 0 \cdot 5^0 = a$$

$$17 = F_1 = a \cdot 5^1 + b \cdot 1 \cdot 5^1 = 5a + 5b$$

Solving these two equations, we get $a=3$ and $b=2/5$

Hence, the final solution is – $F_n = 3 \cdot 5^n + (2/5) \cdot n \cdot 2^n$

Problem 3 Solve the recurrence relation $F_n = 2F_{n-1} - 2F_{n-2}$ where $F_0=1$ and $F_1=3$

Solution The characteristic equation of the recurrence relation is –

$$x^2 - 2x - 2 = 0$$

Hence, the roots are – $x_1=1+i$ and $x_2=1-i$

In polar form,

$$x_1 = r\angle\vartheta \text{ and } x_2 = r\angle(-\vartheta), \text{ where } r=\sqrt{2} \text{ and } \vartheta=\pi/4$$

The roots are imaginary. So, this is in the form of case 3.

Hence, the solution is –

$$F_n = (\sqrt{2})^n (\text{acos}(n \cdot \pi/4) + b\text{sin}(n \cdot \pi/4))$$

$$1 = F_0 = (\sqrt{2})^0 (\text{acos}(0 \cdot \pi/4) + b\text{sin}(0 \cdot \pi/4)) = a$$

$$3 = F_1 = (\sqrt{2})^1 (\text{acos}(1 \cdot \pi/4) + b\text{sin}(1 \cdot \pi/4)) = \sqrt{2}(a/\sqrt{2} + b/\sqrt{2})$$

Solving these two equations we get $a=1$ and $b=2$

Hence, the final solution is –

$$F_n = (\sqrt{2})^n (\cos(n\pi/4) + 2\sin(n\pi/4))$$

Non-Homogeneous Recurrence Relation

A recurrence relation is called non-homogeneous if it is in the form

$$F_n = AF_{n-1} + BF_{n-2} + f(n) \text{ where } f(n) \neq 0$$

Particular solution: A recurrence relation $F_n = AF_{n-1} + BF_{n-2} + f(n)$ where $f(n) \neq 0$ Its associated **homogeneous recurrence relation** is $F_n = AF_{n-1} + BF_{n-2}$

The solution (a_n) of a non-homogeneous recurrence relation has two parts. First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the **particular solution** (a_t)

$$a_n = a_h + a_t$$

To find the **particular solution**, we find an appropriate trial solution.

Let $f(n) = cx^n$; let $x^2 = Ax + B$ be the characteristic equation of the associated homogeneous recurrence relation and let x_1 and x_2 be its roots.

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $a_t = An^2x^n$

Total Solution: The total solution of linear recurrence relations with constant coefficients involves combining homogeneous and particular solution followed by determining constants.

Example

Let a non-homogeneous recurrence relation be $F_n = AF_{n-1} + BF_{n-2} + f(n)$ with characteristic roots $x_1 = 2$ and $x_2 = 5$. Trial solutions for different possible values of $f(n)$ are as follows –

$f(n)$	Trial solutions
4	A
$5 \cdot 2^n$	$An2^n$
$8 \cdot 5^n$	$An5^n$
4^n	$A4^n$
$2n^2 + 3n + 1$	$An^2 + Bn + C$

Problem:

Solve the recurrence relation $F_n = 3F_{n-1} + 10F_{n-2} + 7 \cdot 5^n$ where $F_0 = 4$ and $F_1 = 3$

Solution:

This is a linear non-homogeneous relation, where the associated homogeneous equation is $F_n = 3F_{n-1} + 10F_{n-2}$ and $f(n) = 7 \cdot 5^n$

The characteristic equation of its associated homogeneous relation is –

$$x^2 - 3x - 10 = 0$$

$$\text{Or, } (x-5)(x+2) = 0$$

$$\text{Or, } x_1 = 5 \text{ and } x_2 = -2$$

Hence $a_h = a \cdot 5^n + b \cdot (-2)^n$, where a and b are constants. Since $f(n) = 7 \cdot 5^n$, i.e. of the form $c \cdot x^n$, a reasonable trial solution of a_t will be Anx^n

$$a_t = Anx^n = An5^n$$

After putting the solution in the recurrence relation, we get –

$$An5^n = 3A(n-1)5^{n-1} + 10A(n-2)5^{n-2} + 7 \cdot 5^n$$

Dividing both sides by 5^{n-2} , we get

$$An5^2 = 3A(n-1)5 + 10A(n-2)5^0 + 7 \cdot 5^2$$

$$\text{Or, } 25An = 15An - 15A + 10An - 20A + 175$$

$$\text{Or, } 35A = 175$$

Or, $A=5$

So, $F_n = An5^n = 5n5^n = n5^{n+1}$

The solution of the recurrence relation can be written as –

$$F_n = a_n + a_t = a \cdot 5^n + b \cdot (-2)^n + n5^{n+1}$$

Putting values of $F_0=4$

and $F_1=3$, in the above equation, we get $a=-2$ and $b=6$

Hence, the solution is –

$$F_n = n5^{n+1} + 6 \cdot (-2)^n - 2.5n$$

Generating Functions - Generating Functions represents sequences where each term of a sequence is expressed as a coefficient of a variable x in a formal power series.

Mathematically, for an infinite sequence, say $a_0, a_1, a_2, \dots, a_k, \dots$, the generating function will be –

$$G(x) = a_0 + a_1x + a_2x^2 + \dots + a_kx^k + \dots = \sum a_kx^k \quad (k=0 \text{ to } \infty)$$

Some Areas of Application

Generating functions can be used for the following purposes –

- For solving a variety of counting problems. For example, the number of ways to make change for a Rs. 100 note with the notes of denominations Rs.1, Rs.2, Rs.5, Rs.10, Rs.20 and Rs.50
- For solving recurrence relations
- For proving some of the combinatorial identities
- For finding asymptotic formulae for terms of sequences

Solution by method of generating functions

Problem 1 - What are the generating functions for the sequences $\{a_k\}$ with $a_k=2$ and $a_k=3k$?

Solution When $a_k=2$, generating function, $G(x) = \sum_{k=0}^{\infty} 2x^k = 2 + 2x + 2x^2 + 2x^3 + \dots$

When $a_k=3k$, $G(x) = \sum_{k=0}^{\infty} 3kx^k = 0 + 3x + 6x^2 + 9x^3 + \dots$

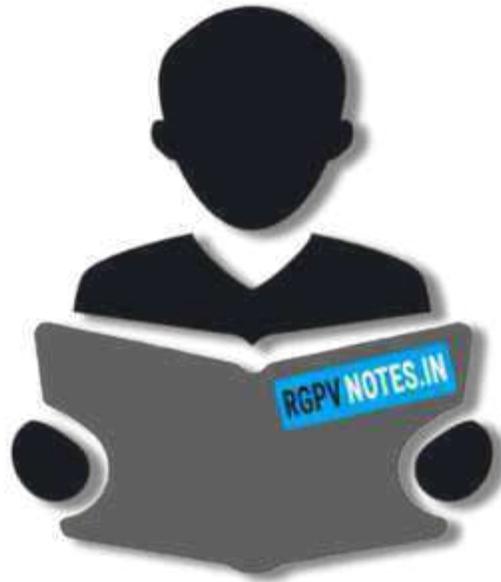
Problem 2 - What is the generating function of the infinite series; 1, 1, 1, 1...?

Solution Here, $a_k=1$, for $0 \leq k \leq \infty$

Hence, $G(x) = 1 + x + x^2 + x^3 + \dots = 1/(1-x)$

Some Useful Generating Functions

- For $a_k = a^k$, $G(x) = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \dots = 1/(1-ax)$
- For $a_k = (k+1)$, $G(x) = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \dots = 1/(1-x)^2$
- For $a_k = c^n$, $G(x) = \sum_{k=0}^{\infty} c^n x^k = 1 + c^n x + c^n x^2 + \dots = 1/(1-c^n x)$
- For $a_k = 1/k!$, $G(x) = \sum_{k=0}^{\infty} x^k/k! = 1 + x + x^2/2! + x^3/3! + \dots = e^x$



RGPVNOTES.IN

We hope you find these notes useful.

You can get previous year question papers at
<https://qp.rgpvnotes.in> .

If you have any queries or you want to submit your
study notes please write us at
rgpvnotes.in@gmail.com



LIKE & FOLLOW US ON FACEBOOK
facebook.com/rgpvnotes.in